# An Axiomatic Characterization of CFMMs and Equivalence to Prediction Markets



Maneesha Papireddygari


Rafael Frongillo


Bo Waggoner

University of Colorado, Boulder


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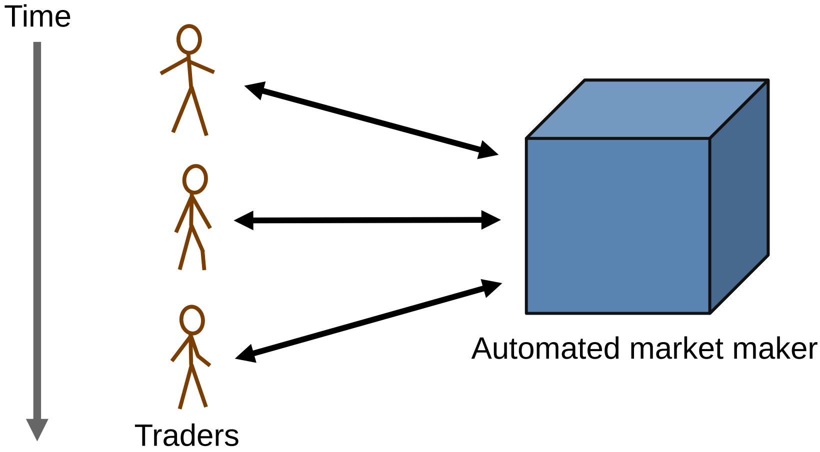
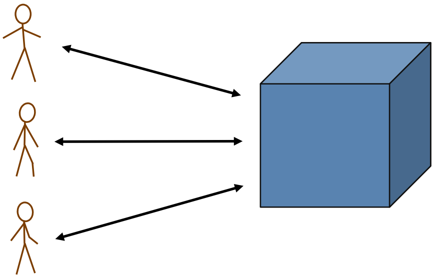
# Automated Market Makers



# AMMs - motivation and summary

- Recent usefulness on blockchains - Uniswap, etc
- Historical cornerstone of prediction markets (including on blockchains)



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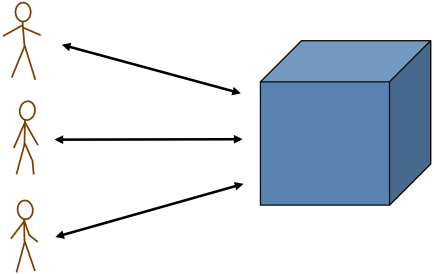

Takeaways from this talk:

- Direct reductions between prediction markets and CFMMs
- Designing for functionality $=$ designing for elicitation

# Outline

1. Constant-Function Market Makers (CFMMs)
   - Market-making axioms
2. Prediction markets
   - Cost-function designs
   - Convex analysis for traders
3. Main results
   - Equivalence of the two market makers
   - Extension to liquidity levels

# Constant-Function Market Makers (CFMMs)

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**Question:** what are good pricing rules?

# Constant-function market makers (CFMMs)

$r$ = *trade*,
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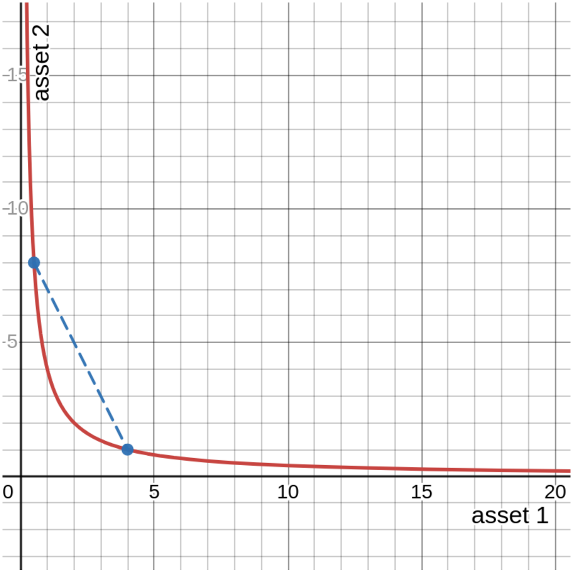
**Primary example:** Constant-product, e.g. Uniswap v2:

$$\varphi(\mathbf{q}) = \left(\prod_{i=1}^{n} \mathbf{q}_i\right)^{1/n}.$$

*e.g. with two assets, $\varphi(q_1, q_2) = \sqrt{q_1 q_2}$.*

# Characterization of CFMMs



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**Proposition:** If and only if an automated market satisfies:

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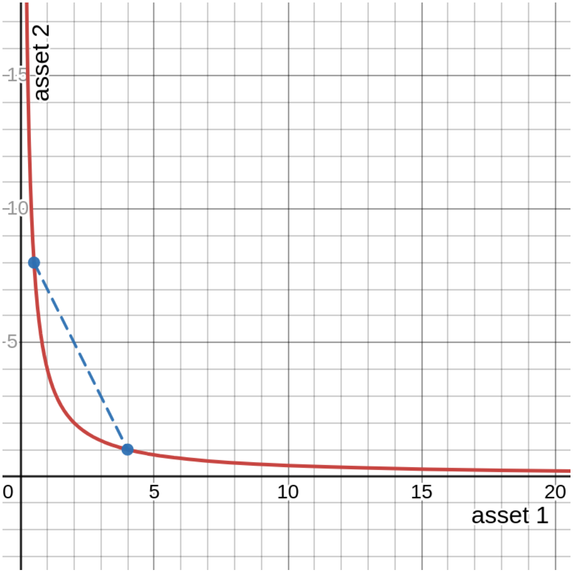

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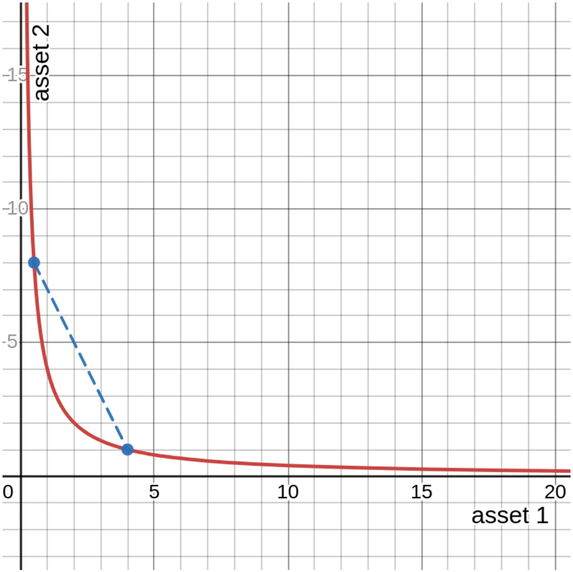

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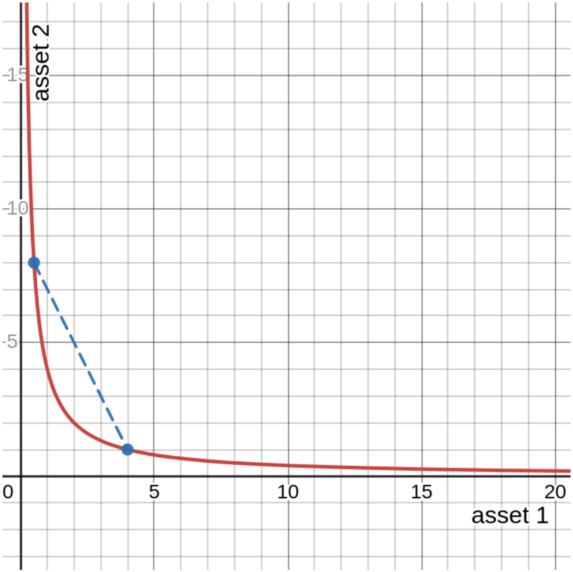

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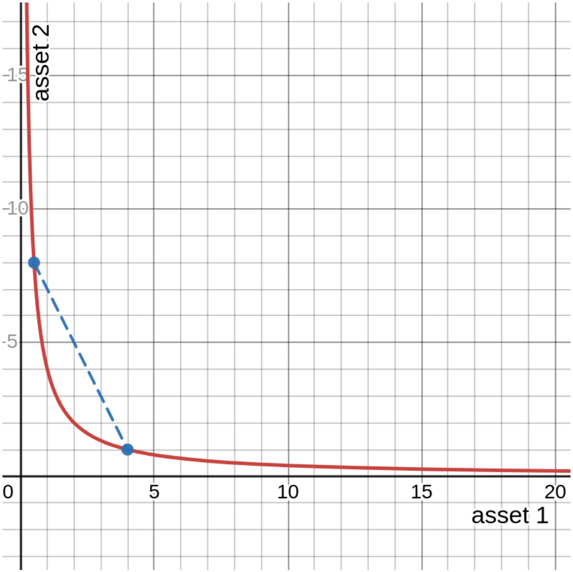

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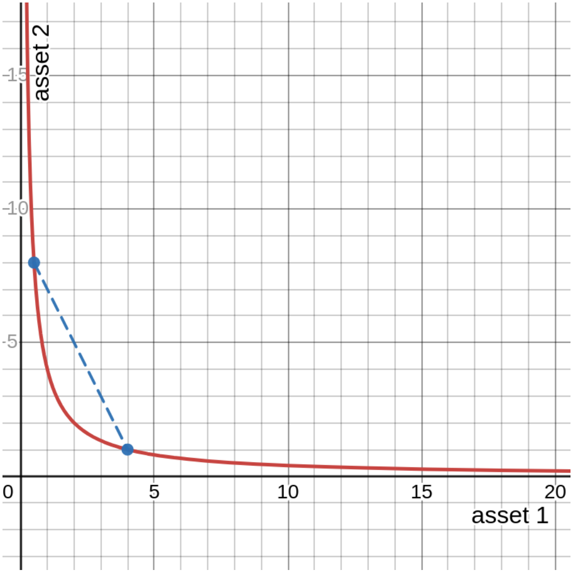

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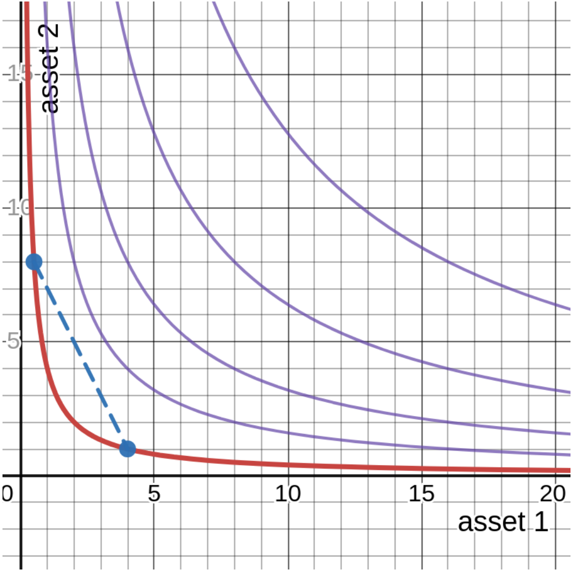
Extension to multiple level sets: **quasiconcave**.



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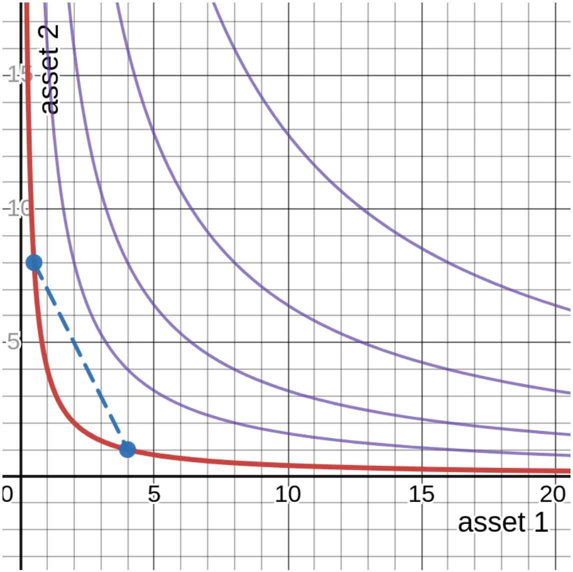
Angeris and Chitra (2020), Angeris et al. (2022), Bichuch and Feinstein (2022): axioms *about CFMMs*; here, deriving CFMM structure.



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**Question:** how to elicit market belief?

*a priori: different design goal than facilitating trade*

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accept any $\mathbf{r} \in \mathbb{R}^n$ and pay $C(\mathbf{q}+\mathbf{r}) - C(\mathbf{q})$ cash.

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- **Characterize** truthful, path-independent markets[2].

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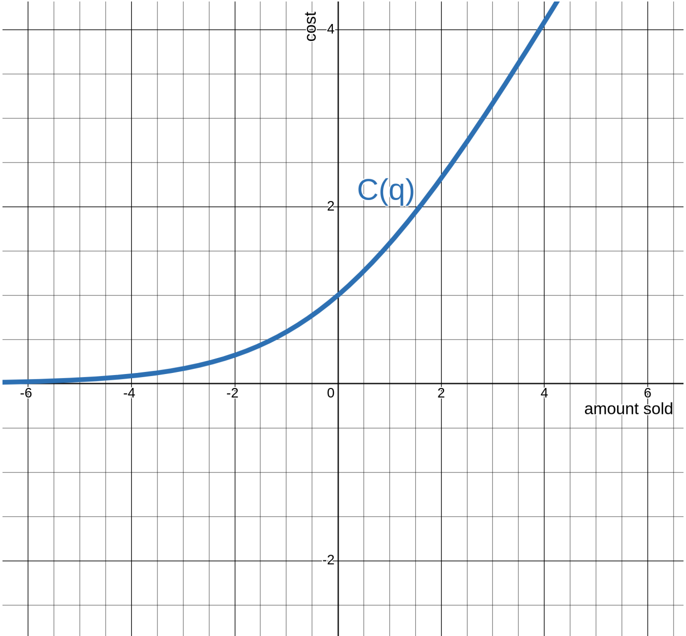
# Cost functions, visually



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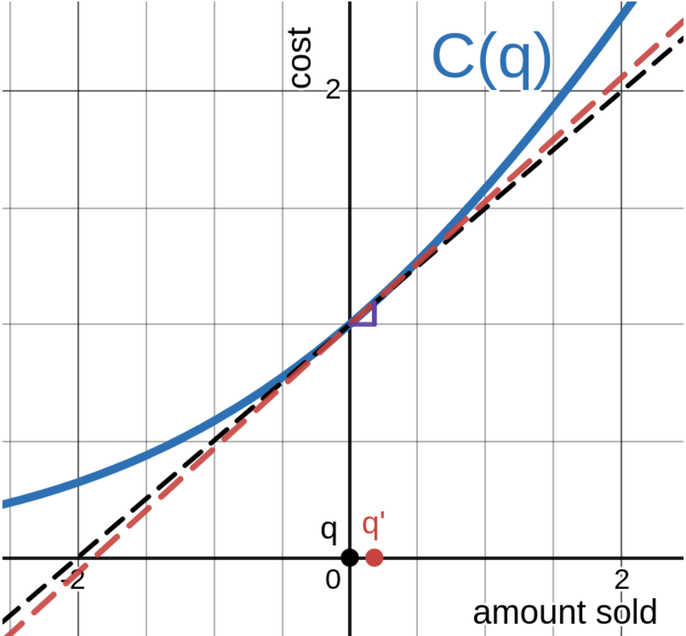

# Market design through convex analysis

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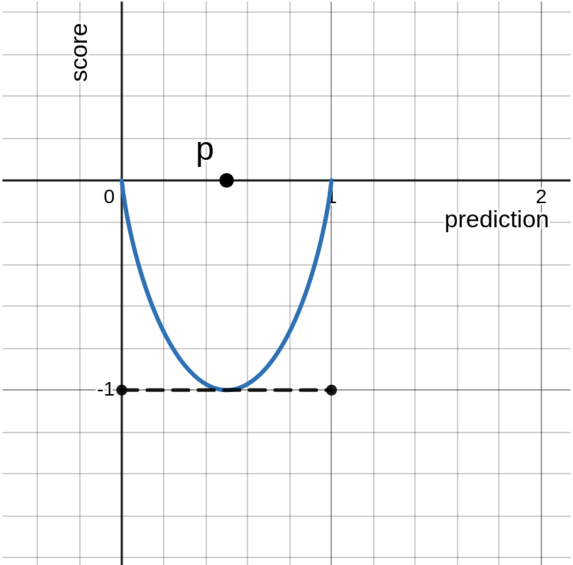


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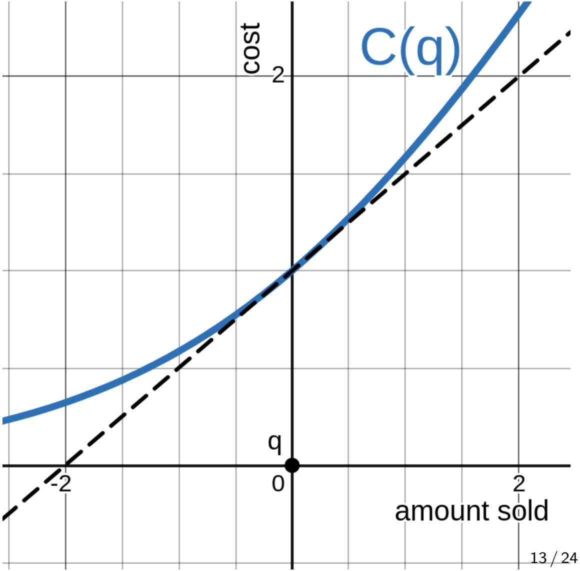

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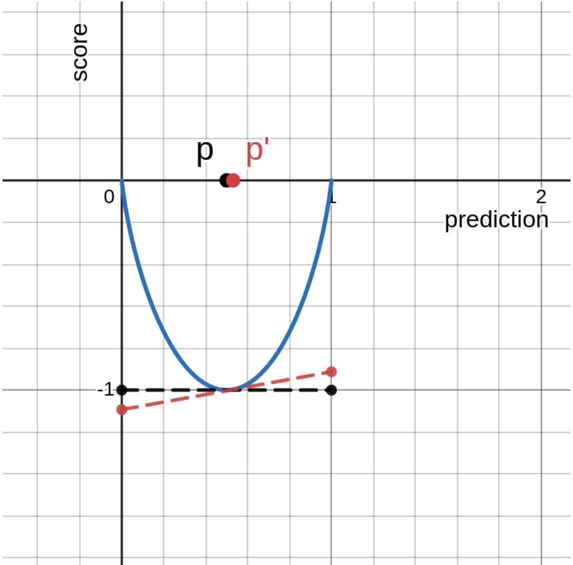


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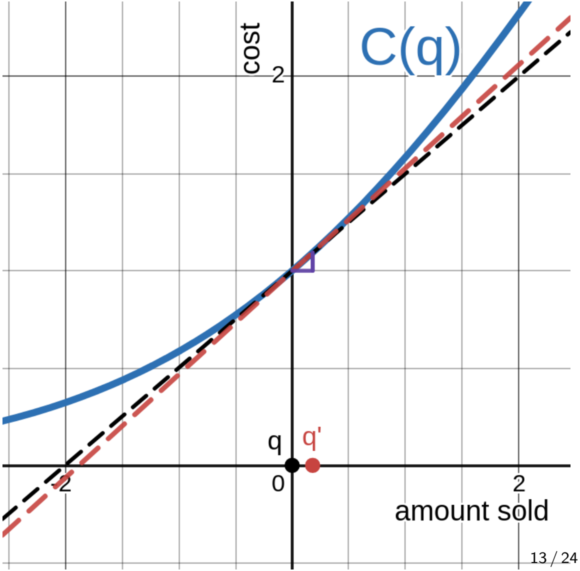

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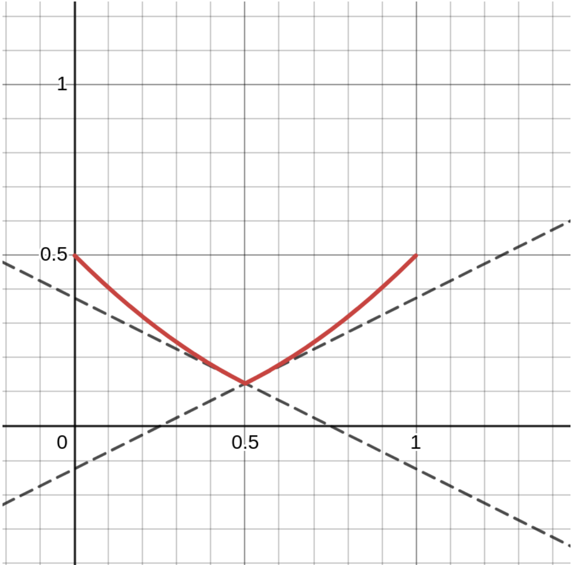


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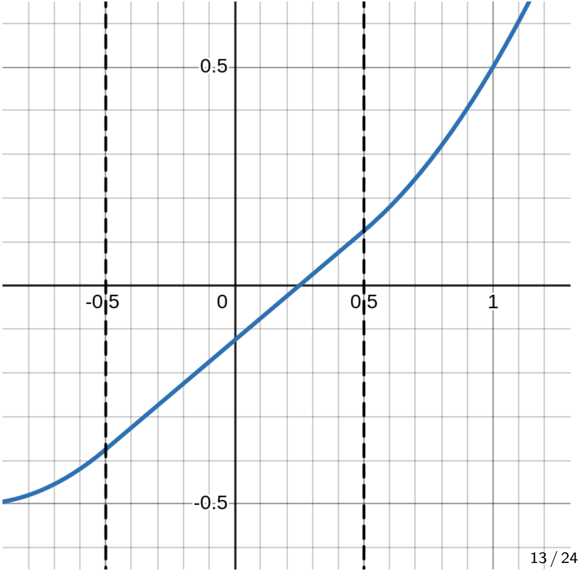

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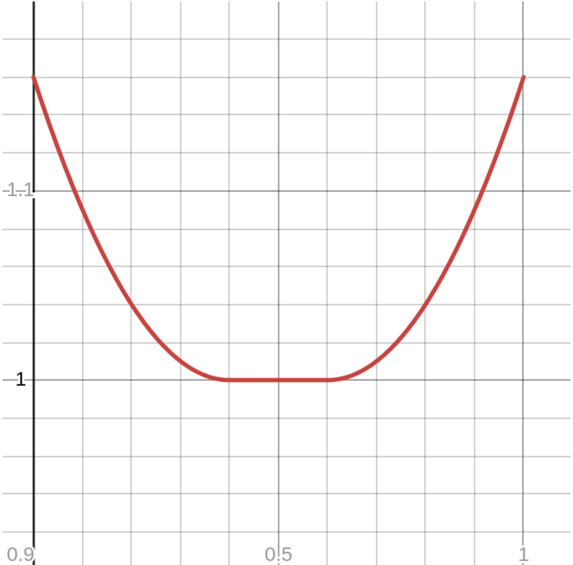


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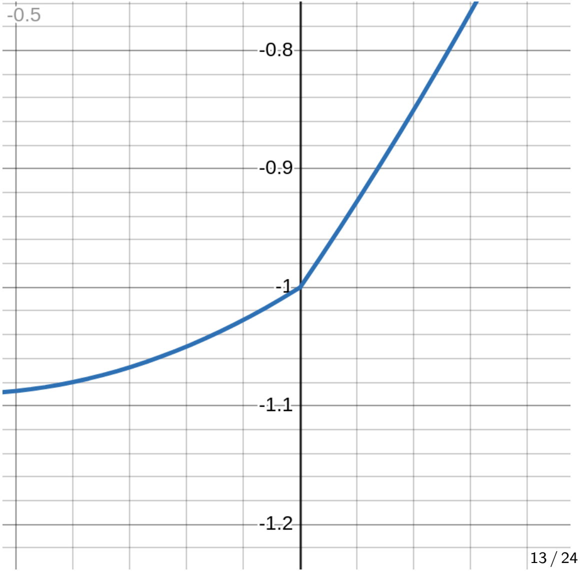

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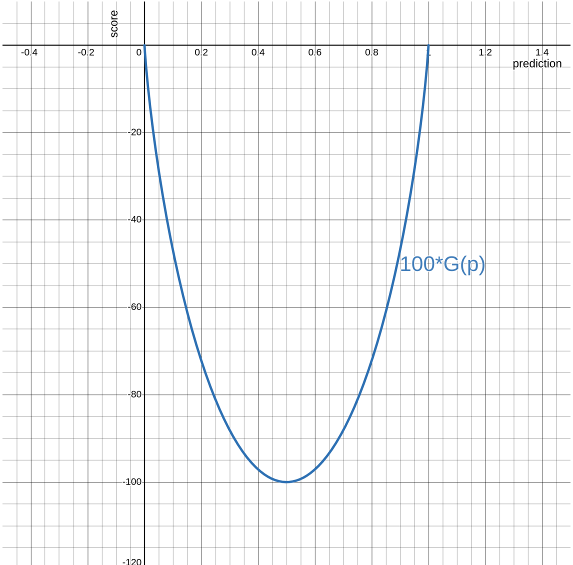


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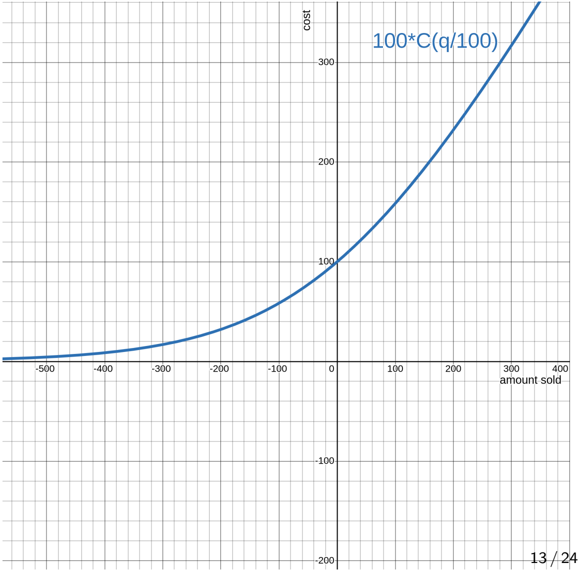

# Main result 1: Equivalence

# Equivalence[3]

**Theorem:** There are reductions between the space of:

- Cost function markets with convex, ones-invariant $C$, and
- CFMMs with concave, increasing $\varphi$

such that the automated markets implemented are the same.$^*$

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*Needs explanation:

- Prediction markets assume cash
- Prediction market assets are specizialized

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Recall: $n+1$ assets (security $1, \ldots,$ security $n$, cash).

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*Proof:* Ones-invariance implies $C(\mathbf{q} + \mathbf{r} - \alpha\mathbb{1}) = C(\mathbf{q})$. $\square$

*And $\varphi(\mathbf{q}) = -C(-\mathbf{q})$ is concave, increasing.*

# $(\Longleftarrow)$

**Theorem:** Given a concave, increasing $\varphi$ and initial reserves $\mathbf{q}_0$, the function

$$C(\mathbf{q}) := \inf \{c \in \mathbb{R} \mid \varphi(c\mathbb{1} - \mathbf{q}) \geq \varphi(\mathbf{q}_0)\}$$

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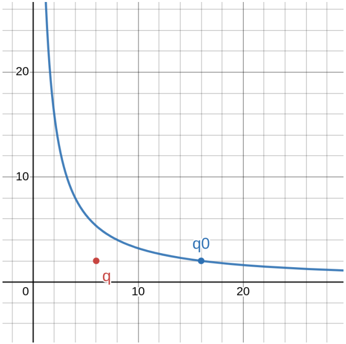

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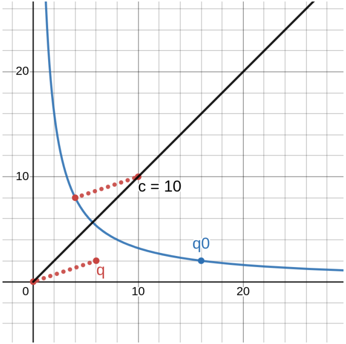

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# Takeaways from equivalence

## 1. Constructions



Logarithmic utility and negative exponential utility are two widely used utility functions that both belong to the HARA utility class. The cost function corresponding to the logarithmic utility function, $u(m) = \log(b+m)$ with $b > 0$, is the implicit function defined by equation (13). If the event only has two outcomes and the market maker's subjective probability estimate is uniform, the explicit cost function for the market maker is

$$C(\vec{q}) = -b + \frac{1}{2}(q_1 + q_2) + \frac{1}{2}\sqrt{4b^2 + (q_1 - q_2)^2}. \quad (14)$$

For the negative exponential utility function, $u(m) = -e^{-\alpha m}$ with $\alpha > 0$, the market maker's cost function is

$$C(\vec{q}) = \frac{1}{\alpha}\log\Big(\sum_j \pi_j e^{\alpha q_j}\Big). \quad (15)$$

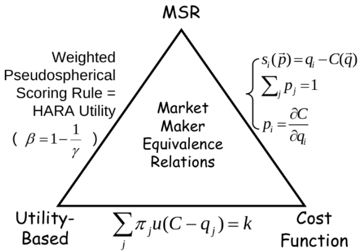
We omit the price functions here, which can be easily obtained by differentiation.

### 5.3 Cost Functions and Market Scoring Rules

which is equivalent to the cost function (15) derived for negative exponential utility market maker by setting $\vec{\pi}$ to be uniform and with some variable substitution, verifying the stated equivalence result in Corollary 4. The equivalent cost function for a MSR market maker with a quadratic scoring rule (2) is

$$C(\vec{q}) = \frac{\sum_j q_j}{N} + \frac{\sum_j q_j^2}{4b} - \frac{(\sum_j q_j)^2}{4Nb} - \frac{b}{N}, \quad (18)$$

Price functions can be obtained by differentiation.



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**scoring rule** $S(p, i) = -b\sqrt{\frac{p_i}{p_j}}$ *Buha (2005), Ben-David and Blais (2020)*

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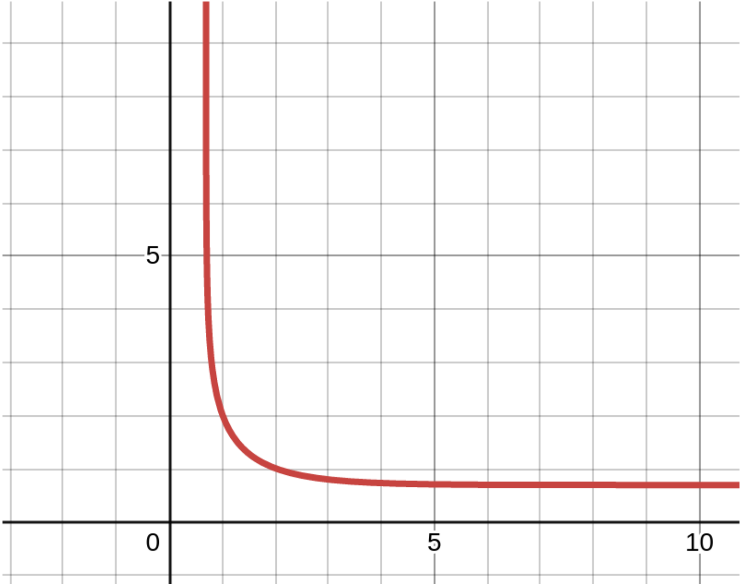
CFMMs elicit **ratios of valuations** *e.g. BTC:ETH $\approx$ 10:1*

# Main result 2: Liquidity levels

# Liquidity levels

Our reduction: prediction market $\rightarrow$ CFMM



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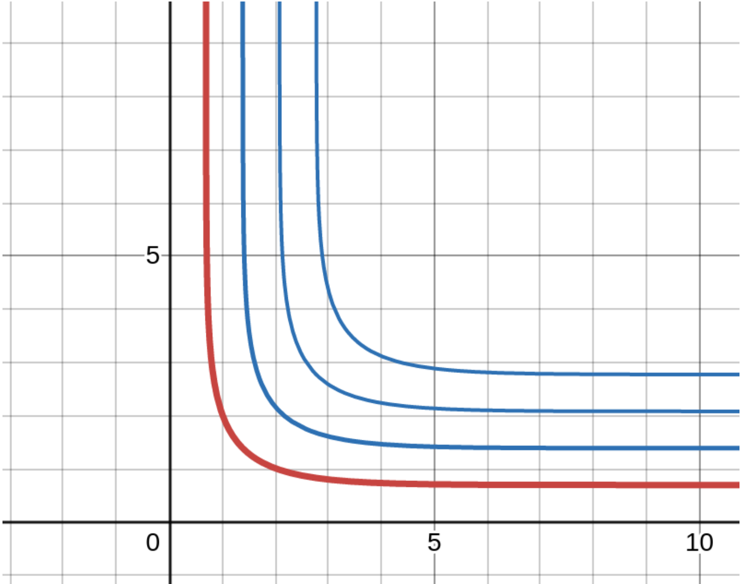

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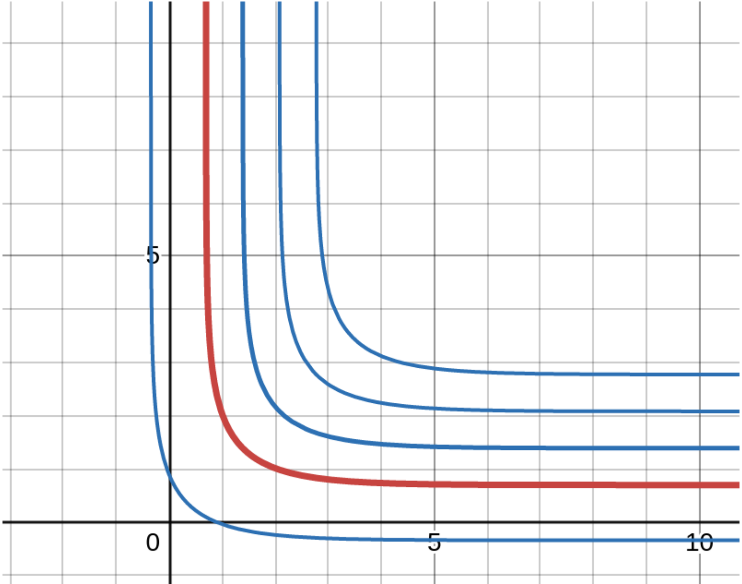

# Liquidity levels - fixed

**Theorem:** Let $C$ be convex and ones-invariant; then

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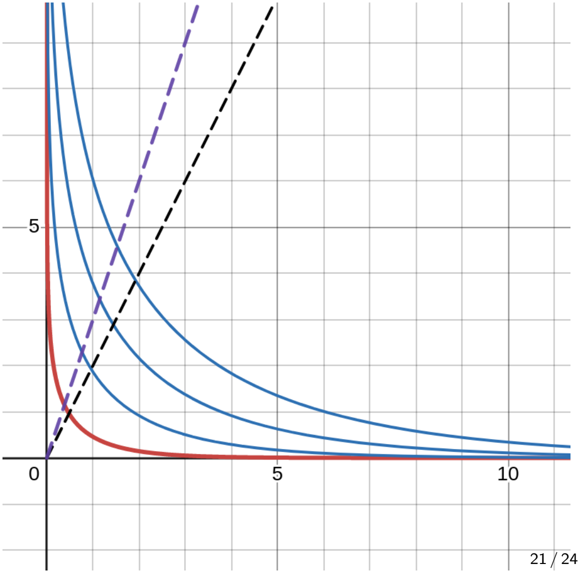
$^*$Trades and exchange rates do not change if all quantities are multiplied by $c > 0$.

**Theorem:** The converse holds.



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- Adaptive liquidity via transaction fees
- Connections with prediction markets:
  - Online learning
  - Assets with negative values or predefined relationships
  - Combinatorial markets?
  - “Arbitrage” reduction

Some prediction market references:


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**Thanks!**